Int. J. Heat Mass Transfer. Vol. 28, No. 1, pp. 101–109, 1985
Printed in Great Britain

0017-9310/85 $3.00 + 0.00
Pergamon Press Ltd.

Unsteady momentum and energy boundary layers in laminar flat plate flow

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(*Received* 16 *January* 1984 *and in revised form* 16 *April* 1984)

Abstract—An approximate analytical solution for the unsteady velocity and temperature fields in laminar flat plate flow is presented. The fluid and the plate are initially at rest and isothermal. The plate is impulsively set into motion at constant velocity, the wall temperature is kept constant at a value different from the initial one. The boundary layer equations have been solved by extending a method previously developed by one of the authors, based on expressing the unknown velocity and temperature profiles by the Taylor formula in terms of a suitable variable and on using differential and integral relations obtained from the balance equations. The solution, obtained in closed form for the first order of approximation, shows a good agreement with the numerical data available in the literature.

The convergence of the method has been checked in some particular cases, by considering higher orders of approximation in the Taylor formula: the third approximation gives results practically coincident with exact ones.

1. INTRODUCTION

IN THIS paper the heat transfer in the unsteady incompressible laminar boundary layer along a flat plate moving at velocity $U(t)$ is studied.

The problem more frequently considered in literature is the impulsive flow at constant velocity, in which the fluid is initially at rest, and the temperature field is uniform. A forced convection energy layer is then produced by an impulsive motion of the plate and by suddenly imposing a constant temperature difference between the plate and the fluid. The momentum and energy boundary layers develop simultaneously, exhibiting three distinct flow regimes. At a fixed position along the plate, initially the two boundary layers are independent of upstream flow history (Rayleigh regime). Asymptotically the flow tends to the steady state (Blasius regime). Between these two regimes there is the transition one. Stewartson (1951) presented an approximate solution for the momentum boundary layer in impulsive flow on a flat plate [1]. Similar results were obtained by Schuh [2], Oudart [3] and Cheng and Elliot [4]. The problem was solved by means of numerical procedures by Dwyer [5], Hall [6] and Dennis [7]. More recently, simultaneous development of the energy and momentum boundary layer has been treated by Watkins for the flat plate [8] and for the Falkner-Skan flows [9].

It is the authors' opinion that a manageable analytical solution is a useful tool in understanding the physics of the problem and in obtaining the asymptotic behaviour of the quantities of interest.

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In this paper the method used is an extension of one developed by one of the authors for the unsteady momentum boundary layer equations [10] and [11]. In particular the method consists of expressing the unknown velocity and temperature profiles by the Taylor formula in terms of a suitable variable and of using differential and integral relations derived from the balance equations. The accuracy of the solutions depends on the number of terms in the Taylor formulas; some previous applications to steady heat transfer problem [12] and to unsteady velocity fields [13] and [14] have shown that even the first term of the Taylor formula (for some problems the second term is needed) provides accurate results.

As a first application to unsteady heat transfer problems, the method has been applied to impulsive flow along a flat plate. The solutions, obtained by the first term of Taylor formulas, show the existence of two distinct regimes for the velocity profiles, the Rayleigh and the Blasius ones, and three regimes for the temperature profiles. The results are in good agreement with the numerical data available in the literature. Moreover, solutions are obtained by adding the second terms in the Taylor formulas. The higher accuracy of the results can be considered as proof of the convergence of the proposed expansion.

Thus we note that the possibility of obtaining a simple and accurate representation of complex thermo-fluid-dynamic fields is given essentially by the presence of singular points in the solution. In fact, by means of the unit function of a suitable argument, the first approximation already characterizes the basic aspects of the field, while the singular points of the higher approximations give a more articulated representation of the solution.

NOMENCLATURE

a_i	$= \int_0^\infty \mathrm{erfc}^i\, x\, dx$
b_{ij}	$= \int_0^\infty \mathrm{erf}^i\, x\, \mathrm{erfc}^j(kx)\, dx$
c_D	= drag coefficient
c_1, c_2	coefficients defined in the equations (3.6)
g_i	functions defined by equation (2.8)
h, k	scaling functions
m_i	functions defined by equation (2.9)
s, s_1	functions defined by equation (3.6)
u, v	dimensionless Cartesian components of the velocity
x, y	dimensionless Cartesian coordinates
t	dimensionless time
z	$= \tau/\xi$ similitude variable
z_v	upper limit of the velocity Rayleigh regime
z_T	upper limit of the temperature Rayleigh regime
Nu	Nusselt number
Pr	Prandtl number
Re	Reynolds number
$\hat{X}, \hat{T}$	functions defined by equation (3.8) characterizing the boundary conditions
T	temperature
U	outer velocity
$\bar{U}$	unit function
Z_i	variables in terms of which are expanded u and ϑ [equations (2.14)–(2.15)]
α, β	functions defined by equations (2.12)
δ_1, δ_2	functions defined by equations (2.13)
η	$= y/h$
ϑ	$= T/T_\infty$
ξ	$= x$
τ	$= t$
$(\ )_i, (\ )_0, (\ )_w, (\ )_\infty$	conditions at $t = 0$, $x = 0$, $y = 0$, $y = \infty$
$(\ )_s$	steady values.

2. BASIC EQUATIONS

The laminar boundary layer equations for plane, incompressible, flat plate unsteady flow, in dimensionless form, are

$$u_x + v_y = 0 \tag{2.1}$$

$$u_t + uu_x + vu_y = U_t + u_{yy} \tag{2.2}$$

$$\theta_t + u\theta_x + v\theta_y = \theta_{yy}/Pr \tag{2.3}$$

where $\theta = T/T_\infty$; T_∞ and U are the temperature and the velocity for the inviscid potential flow.

The following boundary and initial conditions will be considered here

$$u(x, y, 0) = u_i(x, y);\quad u(x, 0, t) = v(x, 0, t) = 0;\quad u(0, y, t) = u_0(y, t) \tag{2.4}$$

$$T(x, y, 0) = T_i(x, y);\quad T(x, 0, t) = T_w(x, t);\quad T(0, y, t) = T_0(y, t). \tag{2.5}$$

Equations (2.1)–(2.3) will be solved by means of an extension of the method, presented in refs. [10] and [11] and based on the following considerations. Let $\xi = x$, $\tau = t$, $\eta = y/h(x, t)$, with $h(x, t)$ an unknown function, and $Z_1(\eta)$ and $Z_2(\eta)$ two known suitable functions, infinitely differentiable and such that $Z_i'(\eta) > 0$, $Z_i(0) = 0$ and $Z_i(\infty) = 1$. Thus, expressing the velocity and temperature profiles by the Taylor formula, one has:

$$u^*(\xi, \eta(Z_1), \tau) = \sum_{i=1}^{n-1} (\partial^i u^*/\partial Z_1^i)_0 Z_1^i/i! + (\partial^n u^*/\partial Z_1^n)_{\bar{Z}_1} Z_1^n/n! \tag{2.6}$$

$$\theta^*(\xi, \eta(Z_2), \tau) = \sum_{i=1}^{m-1} (\partial^i \theta^*/\partial Z_2^i)_0 Z_2^i/i! + (\partial^m \theta^*/\partial Z_2^m)_{\bar{Z}_2} Z_2^m/m! \tag{2.7}$$

where $u^* = u/U$ and $\theta^* = (T - T_w)/(T_\infty - T_w)$ and $0 < \bar{Z}_i < Z$.

In the last terms of equations (2.6) and (2.7), the remainders—the values $\bar{Z}_i(Z_i)$—are unknown. We shall calculate the remainders assuming $\bar{Z}_i(Z_i) \sim \bar{Z}_i(Z_i(\infty)) = Z_{i\infty}$. Thus we evaluate the nth derivative in equation (2.6) and the mth in equation (2.7) in the fixed points $Z_{i\infty}$, $0 < Z_{i\infty} < 1$, as here the remainders are known, whereas the points $\bar{Z}_i$, vary with Z_i. It has to be noted that, by this assumption, equations (2.6) and (2.7) give the exact values for the velocity and temperature at $Z(\eta = 0) = 0$ and $Z(\eta = \infty) = 1$: in fact $u(\eta = 0) = 0$, $u(\eta = \infty) = U$, $T(\eta = 0) = T_w$, $T(\eta = \infty) = T_\infty$.

To calculate the derivatives of u^* and θ^* at $Z_{i\infty}$, let us write the equations (2.6) and (2.7) at $Z_i = 1$:

$$(\partial^n u^*/\partial Z_1^n)_{Z_{1\infty}}/n! = 1 - \sum_{i=1}^{n-1} (\partial^i u^*/\partial Z_1^i)_0/i!$$

$$(\partial^m \theta^*/\partial Z_2^m)_{Z_{2\infty}}/m! = 1 - \sum_{i=1}^{m-1} (\partial^i \theta^*/\partial Z_2^i)_0/i!$$

Hence:

$$u^* = \sum_{i=1}^{n-1} g_i(Z_1^i - Z_1^n) + Z_1^n \tag{2.8}$$

$$\theta^* = \sum_{i=1}^{m-1} m_i(Z_2^i - Z_2^m) + Z_2^m \tag{2.9}$$

where $g_i = (\partial^i u^*/\partial Z_1^i)_0/i!$ and $m_i = (\partial^i \theta^*/\partial Z_2^i)_0/i!$

Integrating the momentum and energy equations with respect to η between 0 and ∞, one obtains two first-order partial differential equations; the remaining equations are obtained differentiating the balance equations with respect to η and evaluating them at $\eta = 0$.

The integration of the momentum and energy equations gives

$$(hU\alpha)_\tau + (hU^2\beta)_\xi = u_{\eta,0}/h \quad (2.10)$$

$$(hT_\infty\delta_2)_\tau + (hUT_\infty\delta_1)_\xi = (T_\infty\theta)_{\eta,0}/hPr \quad (2.11)$$

where

$$\alpha(\xi,\tau) = \int_0^\infty (1-u/U)\,d\eta;$$

$$\beta(\xi,\tau) = \int_0^\infty (u/U)(1-u/U)\,d\eta \quad (2.12)$$

$$\delta_2(\xi,\tau) = \int_0^\infty (1-\theta)\,d\eta;$$

$$\delta_1 = \int_0^\infty (u/U)(1-\theta)\,d\eta \quad (2.13)$$

A convenient choice of $Z_1(\eta)$ and $Z_2(\eta)$ is, of course, a delicate problem.

The asymptotic behaviour of the exact solution suggests the choice of $Z_1(\eta)$ and $Z_2(\eta)$ as the error functions:

$$Z_1 = \operatorname{erf}\eta \quad (2.14)$$

$$Z_2 = \operatorname{erf}(k\eta). \quad (2.15)$$

The unknown functions $h(\xi,\tau)$ and $h/k(\xi,\tau)$ are a measure of the momentum and energy boundary layer, respectively.

3. FIRST ORDER APPROXIMATION

If the Taylor formulas are both restricted to the linear terms, the velocity and temperature profiles are given by

$$u(\xi,\eta,\tau)/U(\xi,\tau) = Z_1(\eta) = \operatorname{erf}\eta \quad (3.1)$$

$$\theta^*(\xi,\eta,\tau) = Z_2(\eta) = \operatorname{erf}(k\eta). \quad (3.2)$$

The two unknown scaling functions $h(\xi,\tau)$ and $k(\xi,\tau)$ are determined by the two first-order partial differential equations (2.10) and (2.11).

The $h(\xi,\tau)$ function (and then $k(\xi,\tau)$) can be obtained by the Lagrange method, see refs. [10] and [11]. Equations (2.10) and (2.11) can be written in the form:

$$f_1 H_\xi + f_2 H_\tau = f_3 \quad (3.3)$$

where $f_i = f_i(\xi,\tau,H)$. The associated characteristic system is

$$d\tau/f_1 = d\xi/f_2 = dH/f_3. \quad (3.4)$$

If $F_1(\tau,\xi,H) = $ const. and $F_2(\tau,\xi,H) = $ const. are two independent integrals of equations (3.4), the general solution can be written as $F_2 = F(F_1)$, where F is determined through the boundary conditions.

The momentum equation (equation (2.10)) can be solved independently of the energy one (equation (2.11)). Let $H(\xi,\tau) = h^2(\xi,\tau)U^2(\tau)$.

For the assumed velocity profile, α and β are constant and the associated characteristic system can be easily solved; in fact two independent integrals of system (3.4) are

$$\xi - s(\tau) = \text{const.}; \quad h^2U^2 - s_1(\tau) = \text{const.} \quad (3.5)$$

where

$$s(\tau) = c_1\int_0^\tau U(\tau)\,d\tau; \quad s_1(\tau) = c_2\int_0^\tau U^2(\tau)\,d\tau;$$

$$c_1 = \beta/\alpha, \quad c_2 = 4/\alpha\pi^{1/2}. \quad (3.6)$$

Therefore the general solution of equation (2.10) is

$$U^2(\tau)h^2(\xi,\tau) = s_1(\tau) + F(\xi - s(\tau)). \quad (3.7)$$

The function F is obtained through the following boundary conditions on the ξ and τ axes

$$h^2(0,\tau) = \hat{T}(\tau); \quad h^2(\xi,0) = \hat{X}(\tau). \quad (3.8)$$

It has to be noted that the uniform velocity profile gives $h^2 = 0$; in fact $h^2 = 0$ implies $\eta = \infty$ and $Z_1(\infty) = 1$.

The solution of equation (2.10), with the boundary conditions (3.8), is

$$U^2(\tau)h^2(\xi,\tau) = s_1(\tau) + \hat{X}(\xi - s)U^2(0) + \bar{U}(s-\xi)[f(s-\xi) - \hat{X}(\xi - s)U^2(0)] \quad (3.9)$$

where $\bar{U}(z)$ is the unit functions ($\bar{U}(z>0) = 1$) and

$$f(s) = U^2(\tau(s)\hat{T}[\tau(s)] - s_1[\tau(s)]. \quad (3.10)$$

4. VELOCITY PROFILES

4.1. *First approximation*

In this section some examples of the application of the method are presented. The case of uniform velocity profiles ($\hat{X} = \hat{T} = 0$), is considered. A first example is given by the impulsive flow starting from rest.

From equation (3.9), the scaling function $h(\xi,\tau)$ is given by:

$$h^2 = 4\tau - 4\xi(\tau/\xi - 1/c_1)\bar{U}(\tau/\xi - 1/c_1) \quad (4.1)$$

The solution (4.1) shows two different behaviours depending on the value of the variable $z = \tau/\xi$. This variable, being $U(\tau) = 1$, can be interpreted as a dimensionless time. The reference time $\tau_r = \int_0^\xi d\xi/U(\tau)$ represents the time interval for the leading edge disturbance travelling to the local inviscid velocity U, reaches the abscissa ξ: for the impulsive flow $\tau_r = \xi$.

Equation (4.1) can be rewritten in the form

$$h^2 = 4\xi(z - (z - z_v)\bar{U}(z - z_v)) \quad (4.2)$$

where $z_v = 1/c_1 = 2.41$ is the upper limit of the

Rayleigh regime. Therefore this approximate solution shows two different regimes: the Rayleigh regime for which the flow is independent of the leading edge effects, until $z < z_v$; and the Blasius steady regime, for $z > z_v$.

Region $z < z_v$. For $z < z_v$, the Rayleigh regime, the solution is $h^2 = 4\xi z = 4\tau$ and it leads to $u = U$ erf $(y/2t^{1/2})$, the exact initial solution that holds for $z \to 0$.

Region $z > z_v$. For $z > z_v$, the Blasius regime, the steady solution is $h^2 = 4z_v\xi$.

The drag coefficient C_D is given by

$$C_D Re_x^{1/2} = 2x^{1/2}/h\pi^{1/2}. \quad (4.3)$$

In Fig. 1 $C_D Re_x^{1/2}$ vs z is drawn. The first approximation solution exhibits the same behaviour as the numerical one of ref. [9] and presents the maximum for the error (less than 10%) at infinity; the steady value is reached at $z = 2.41$, the exact abscissa is nearly 4.

The approximate solution of Stewartson [1], obtained by the classical integral method with $u/U = \sin(\pi y/2\delta)$, predicts the same behaviour as our solution: a Rayleigh regime (the solution is slightly different from the exact one) for $z < 2.65$, and an accurate Blasius regime (0.328 instead of 0.3321) for $z > 2.65$.

As a second example we consider the outer flow $U = \tau^n$.

From equations (3.6) one has $s = c_1\tau^{n+1}/(n+1)$; $s_1 = c_2\tau^{2n+1}/(2n+1)$. Equation (3.9) leads to

$$h^2 = c_2\xi^{n_2}[z^{n_2/n_1} - (z - n_1/c_1)^{n_2/n_1}\bar{U}(z - n_1/c_1)]/n_2$$

where $z = \tau^{n_1}/\xi$, $n_1 = n+1$, $n_2 = 2n+1$. For $n = 0$, i.e. when $U = 1$, one obtains the solution given by (4.1).

A third example is given by $U = 1 - \exp(-a\tau)$; for $a \to \infty$, $U \to 1$ which is the starting flow considered in the first example. From equations (3.6) one has

$$s = c_1(\tau - U/a);$$

$$s_1 = c_2\{\tau - 2U/a + [1 - \exp(-2a\tau)]/2a\}.$$

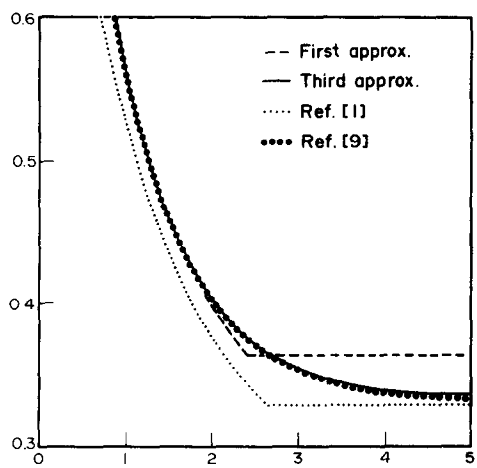

FIG. 1. Drag coefficient vs z, for the impulsive flow.

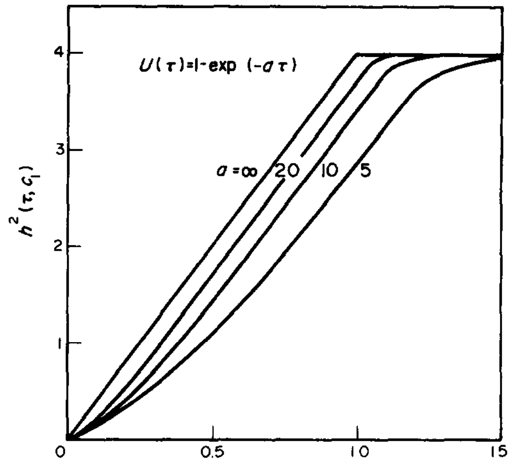

FIG. 2. Velocity scaling function h for $U = 1 - \exp(-a\tau)$.

Equation (3.9) leads to $U^2h^2 = s_1 + \bar{U}(\xi - s)f(s - \xi)$ where $f(s) = -s_1[\tau(s)]$. Thus this solution is obtained in an implicit form, because the inverse function $\tau(s)$ cannot be expressed by an elementary function. However, it is possible to write its series expansion (Appendix A). One has

$$\tau = \sum_{i=1}^{\infty} A_i s^{i/2},$$

where $A_1 = (2/ac_1)^{1/2}$, $A_2 = 1/3c_1$, $A_3 = (a/182c_1^3)^{1/2}$. In Fig. 2 the function h^2 vs z is drawn for several values of a: the curves present an S shape and tend to the limiting case of the impulsive flow for $a \to \infty$.

4.2. *Higher approximation*

The solutions obtained by the first term of the Taylor formula show a reasonable degree of accuracy. However, it is possible to obtain a better approximation by adding more terms in the Taylor formula. This procedure will be applied to the impulsive flow.

In this case ($U = 1$) the second approximation gives a vanishing contribution; therefore we consider the third approximation. From equation (2.8) with $n = 3$ one has:

$$u^* = g(Z_1 - Z_1^3) + Z_1^3.$$

The two unknowns, h and g are determined from equations (2.10) and (2.2) evaluated at $y = 0$. In terms of the unknowns H ($H = U^2h^2$) and g, these equations can be written as:

$$\alpha H_\tau + \beta H_\xi + 2H(\alpha_g g_\tau + \beta_g g_\xi) = 2\gamma g;$$

$$gH_\tau - 2Hg_\tau = (4 - 2C)g - 2C$$

where $\gamma = Z_{1\eta,0}$ and $C = 6\gamma^2$. This system of two partial differential equations becomes an ordinary one by putting $z = \tau/\xi$, $x = \xi$ and $H = xf(z)$, $g = g(z)$. Moreover, by simple manipulation, it can be written in the following normal form

$$f'D(z,g) = N(z,f,g);$$

$$2fDg' = 2CD + g[N - 2D(2+C)] \quad (4.4)$$

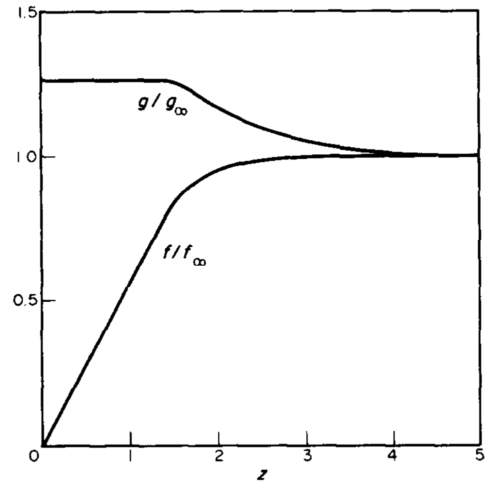

FIG. 3. Velocity functions for the third approximation for the impulsive flow.

where: $\alpha = \alpha_0 - \alpha_1$; $\beta = \beta_0 + \beta_1 g - \beta_2 g^2$; $\alpha_0 = 0.9379$, $\alpha_1 = 0.3737$, $\beta_0 = 0.2378$, $\beta_1 = 0.0925$, $\beta_2 = 0.0966$, $\gamma = 1.1284$; $D = \alpha - \beta z - z[\alpha_1 + z(\beta_1 - 2\beta_2 g)]$; $N = 2\gamma g - f\beta + (\alpha_g - z\beta_g)[2g(2+C) - 2C]$.

This system presents singular points when $f = 0$ or $D = 0$. The first singularity occurs at $z = 0$; in fact the initial conditions require $f = 0$. The initial conditions for g is $g(0) = 1$, in fact, to obtain a finite value of g' at $z = 0$, the right-hand side of the second equation (4.4) needs to vanish.

The functions f and g at $z = 0$ are continuous and are actually the solution of the system (4.4) with initial conditions $f(0) = 0$, $g(0) = 1$, if $f = 4z$, $g = 1$. It corresponds to $u = \mathrm{erf}(y/2t^{1/2})$, i.e. to exact initial solution.

Such a solution holds until the second singular point $z_v = 1.446$, given by $D = 0$. Here the solution assumes a new behaviour: it is possible to show that the functions f and g at $z = z_v$ are continuous with their first derivatives while their second derivatives are infinite. To the right of z_v the solution will be the one that leads to the asymptotic values of f and g, f_∞ and g_∞.

These latter values can be determined *a priori* from the steady solution of the system (4.4): one finds $f_\infty = 7.1416$, $g_\infty = 0.7925$. These asymptotic values give $u_{y,0} = 0.3346\ x^{1/2}$, i.e. a value very close to the exact ones ($0.3321\ x^{1/2}$).

The functions f and g are given in Fig. 3. In Fig. 1, $C_D Re_x^{1/2}$ for the first and the third approximations are compared with the results of refs. [1] and [9]. Some values of $C_D Re_x^{1/2}$ are listed in the following table.

z	0.1	0.5	1	2	4	∞
Third approx.	1.784	0.7979	0.5642	0.4047	0.3359	0.3346
Ref. [9]	1.784	0.8003	0.5622	0.4023	0.3346	0.3321

This table shows that the accuracy of the third approximation is very satisfactory. The third approximation not only coincides with the exact initial solution for $z \to 0$ but also represents well the transition regime. Therefore the present results are very close to that of ref. [9] in all the regions.

This comparison implies that the suggested expansion in terms of z is convergent.

5. TEMPERATURE PROFILES FOR THE IMPULSIVE FLOW

5.1. *First approximation*

In this section we consider both the velocity and the temperature profiles expressed by one term of the Taylor formula.

The temperature field can be obtained in the same way as previously shown for the velocity field. However, we prefer to reduce equation (2.11) to an ordinary one in terms of the variables z and x. The function k is $k = k(z)$ and by putting $k^2 = K$, one has

$$f(\delta_{2k} - z\delta_{1k})K' = 4K/Pr\pi^{1/2} - fk\delta_1 + kf'(z\delta_1 - \delta_2) \quad (5.1)$$

where $\pi^{1/2}\delta_1 = (K+1)^{1/2}/k - 1$; $\pi^{1/2}\delta_2 = 1/k$.

One needs to distinguish the two regions $z < z_v$ and $z > z_v$.

Region $z < z_v$. For $z < z_v$, $f = 4z$; therefore equation (5.1) becomes:

$$z[z/(1+K)^{1/2} - 1]K' = K(K - Pr)/Pr. \quad (5.2)$$

This equation also presents singular points. At $z = 0$ the value of k is evaluated by the condition that $k_z(0)$ is finite; thus $K(0) = Pr$. The solution of equation (5.2) is then $K = Pr$ and holds until $z = z_T = (1+Pr)^{1/2}$ (the second singular point); this point represents the upper limit of the Rayleigh region for the temperature field.

For $z_v < z_T$, i.e. for $Pr > 4.83$, the solution is $K = Pr$.

For $z_v > z_T$, i.e. for $Pr < 4.83$, one has $K = Pr$ for $z < z_T$, while for $z_T < z < z_v$, the solution of equation (5.2) with $k(z_v) = k_s$ (where k_s is the steady value of k) is

$$1/z = (1 - Pr/K)(A + B \log[(w-a)/(w+a)] + 2aBw/K$$

where $B = Pr/c_1$; $A = K_s/(K_s - Pr) - B \log[(w_s - a)/(w_s + a)] - 2aBw_s/(K - Pr)$; $a^2 = 1 + Pr$; $w^2 = 1 + K$; $w_s^2 = 1 + K_s$.

The function k at the singular point z_T is continuous for $Pr < 1$ and presents a jump for $Pr > 1$.

Region $z > z_v$. For $z > z_v$, $f = 4z_v$; therefore equation (5.1) becomes:

$$(\delta_{2k} - z\delta_{1k})K' = K/Prz_v\tau^{1/2} - k\delta_1. \quad (5.3)$$

To associate a condition to this equation, one needs to know k_s. By equating to zero the right-hand side of equation (5.3) one has:

$$(K_s + 1)^{1/2} - k_s = c_1 K_s/Pr. \quad (5.4)$$

From this equation it follows

$$K_s(Pr < 1) > Pr; \quad K_s(Pr = 1) = Pr;$$
$$K_s(Pr > 1) < Pr;$$
$$K_s(Pr \ll 1) \sim 2.4\, Pr; \quad K_s(Pr \gg 1) \sim Pr^{2/3}.$$

For $z_T < z_v(Pr < 4.83)$ the solution is $K = K_s$.

For $z_T > z_v(Pr > 4.83)$ equation (5.3) becomes linear by assuming K as independent variable and can be solved by quadratures with the initial condition at $z = z_v$ given by the previous solution. Also in this case, k presents a jump at $z = z_T$ in order to reach the steady value.

Finally for $Pr = 1$ one has always $K = 1$.

In this way the three different regimes for the temperature field, the Rayleigh, the transition and the Blasius, are fully determined by the previous solutions.

The Nusselt number is given by: $Nu_x Re_x^{-1/2} = 2x^{1/2}k/h\pi^{1/2}$. In Fig. 4 $Nu_x Re_x^{-1/2}$ vs z is given. The present results, compared with the numerical data of ref. [9], show: (i) for $z < 1$ the numerical solution is recovered; (ii) for $1 < z < 1/c_1$ the two curves practically coincide; (iii) the steady value is slightly different from the exact one and it is reached at a definite time, at which the numerical values are close to the asymptotic ones. It has to be noted that the numerical solutions show, for the Nusselt number, two different behaviours. For $Pr < 1$ the curve is rather smooth throughout the whole field; for $Pr > 1$ there is a strong variation around $z \sim 3$. The present solutions imitate these behaviours and, in particular, give a jump in correspondence of the strong variations.

5.2. *Higher approximation*

The accuracy of the solution of the temperature field can also be increased by adding more terms in the Taylor formula. We consider the third approximation both for the velocity field and for the temperature. The temperature profile by means of equation (2.9) with $m = 3$ is:

$$\vartheta^* = m_1 \operatorname{erfc} k\eta + m_2 \operatorname{erfc}^2 k\eta + (1 - m_1 - m_2) \operatorname{erfc}^3 k\eta. \quad (5.5)$$

From equation (2.3) evaluated at $\eta = 0$ one has $\vartheta_{\eta\eta,0} = 0$; it follows that $m_2 = 1.5(1 - m_1)$. Thus one has two unknowns, k and m_1, that can be determined by differentiating equations (2.11) and (2.3) with respect to η and evaluating them at $\eta = 0$; this second equation can be written as:

$$h^2 k m_{1\tau} - h^2 k_\tau (3 - m_1) + h h_\tau (3 - m_1) k = [3 - m_1 + 3\gamma^2(1 - m_1)]k^3/Pr. \quad (5.6)$$

By putting $x = \xi$, $z = \tau/\xi$, $h^2 = xf(z)$, $m_1 = G(z)$, $k^2(z) = K$, and taking into account the result of section 4.2, equations (2.11) and (5.6) can be reduced to the following ordinary system:

$$2fKG' = (3 - G)K'f - (3 - G)f' + 2K[3 - G + 3\gamma^2(1 - G)]/Pr \quad (5.7)$$

$$K' = N_1/D_1 \quad (5.8)$$

where:

$$D_1 = f[(3 - G)(A_2 - zkR_G) - (zKR_k + A_1 + A_2 G)]$$

$$N_1 = \gamma K^2(3 - G)/Pr + 2zKfkR_g g' - KkR(f - zf') - f'K(A_1 + A_2 G) + 2(A_2 - zkR_G)(3 - G)Kf'/2 - 2K^2[3 - G + 3\gamma^2(1 - G)]$$

and:

$$a_i = \int_0^\infty \operatorname{erfc}^i z \, dz; \; A_1 = (1 - A_2);$$

$$\gamma A_2 = a_1 - 3a_2/2 + a_3/2$$

$$\gamma b_{ij}(k) = \int_0^\infty \operatorname{erf}^i z \operatorname{erfc}^j kz \, dz$$

$$\gamma R = \int_0^\infty u\vartheta^* \, d\eta = gGb_{11} + 1.5g(1 - G)b_{12} - (1 - G)gb_{13}/2 + (1 - g)Gb_{31} + 1.5(1 - G) \times (1 - g)b_{32} - (1 - g)(1 - G)b_{33}/2.$$

The system (5.7)–(5.8) presents two singular points due to the velocity and temperature equations: $z_v = 1.4459$ (given by the equation $D = 0$) and z_T (given by the equation $D_1 = 0$).

This latter singular point depends on the Prandtl number. For $Pr < 2.45$ it is $z_T < z_v$; for $Pr > 2.45$, it is $z_T > z_v$.

Region $z < z_v$. For $z < z_v$, $f = 4z$, $g = 1$; therefore $K = Pr$ and $G = 1$ until $z = z_T$; z_T is given by the equation $D_1 = 0$ and $z_T = (2A_2 - 1)/(KR_k + 2kR_G)$. Only for $Pr < 2.45$ is z_T in this region. The functions K and G do not present discontinuity in the singular point z_T for $Pr < 1$, while for $1 < Pr < 2.45$ vanishing discontinuity of the first kind is present. Also in this case for $z > z_T$ the

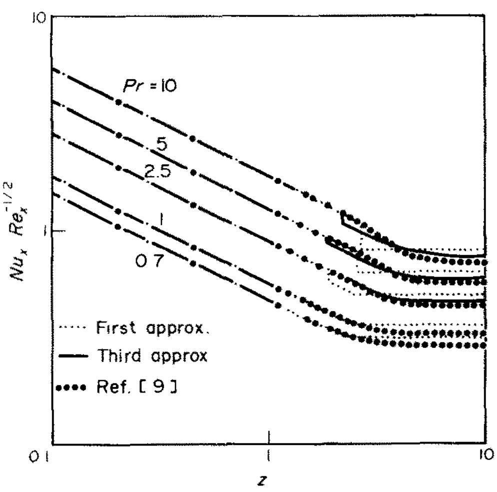

FIG. 4. Nusselt number vs z for several values of the Prandtl number, for the impulsive flow.

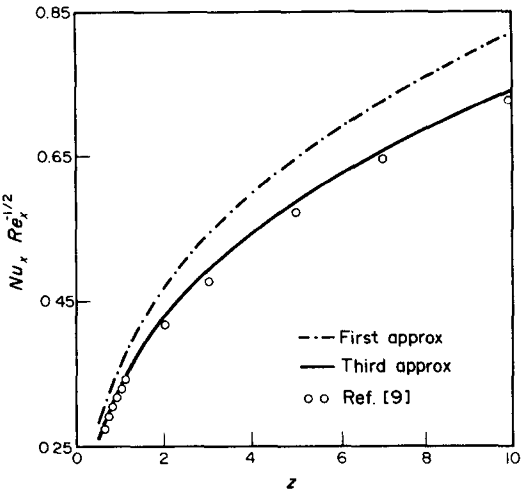

FIG. 5. Asymptotic values of Nusselt number for several values of Prandtl number for the impulsive flow.

solution will be the one that gives the asymptotic values of K and G, $K_{\infty}(Pr)$ and $G_{\infty} = 3 - g_{\infty} = 1.41$. These values can be evaluated *a priori* from the steady solution.

For $Pr > 2.45$ the solution will be $K = Pr$ and $G = 1$ for any z.

Region $z > z_v$. For $z > z_v$ the Rayleigh velocity solution $f = 4z$, $g = 1$ does not hold.

For $Pr < 2.45$ the solution is continuous for any z; while for $Pr > 2.45$, z_T is in this region, and K and G present a jump at z_T whose value, smaller than the corresponding one in the first approximation, increases as Pr increases.

In Fig. 5 the steady values of the Nusselt number for the first and third approximations are compared with the exact numerical ones for several values of the Prandtl number. As one can see, the accuracy of the third approximation is increased with respect to the first one and the agreement with the exact results is very satisfactory. Figure 4 (in which the Nusselt number is plotted vs z for several values of the Prandtl number) shows the same behaviour. The agreement with the numerical results of ref. [9] is thus obtained in all the field.

In the following table, the values of $Nu_x Re_x^{-1/2}$ are compared with the results of ref. [9] for $Pr = 0.7$.

z	0.1	0.5	0.8	1	2	4	∞
Third approx.	1.493	0.668	0.528	0.472	0.358	0.294	0.294
Ref. [9]	1.499	0.669	0.527	0.472	0.344	0.293	0.293

This result enables one to think that even for the temperature field the suggested expansion is convergent.

Thus we note that in correspondence of the strong variation of Nu, the first approximation gives a jump after which it reaches the steady values; the third one again gives a jump, but smaller than the previous one, and its smooth decreasing behaviour fits the numerical results very well.

6. CONCLUDING REMARKS

In this paper the boundary layer equations governing the laminar flow along a flat plate, have been solved, with particular reference to the impulsive flow, obtaining approximate analytical expressions for the velocity and temperature profiles.

The unknown velocity and temperature profiles have been expressed by the Taylor formula in terms of a suitable variable; the evaluation of the remainders are discussed in ref. [11]. The accuracy of the solution increases with the order of the Taylor formula. The first approximation presents an error less than 10%; a second term in the Taylor formula greatly reduces this error.

The compact form in which the results are obtained is due to the presence, in the solution, of singularities that emphasize the behaviour of the quantities of interest with respect to the coordinates and to the Prandtl number.

The drag coefficient and the Nusselt number have been discussed. In the impulsive flow the first approximation imitates the exact solution in the Rayleigh regime and presents a jump, for the Nusselt number, corresponding to a strong variation in the numerical results. The higher approximation also imitates the transition regime for the velocity field. Both drag coefficient and Nusselt number show great accuracy throughout the field; the steady asymptotic values appear particularly accurate.

Acknowledgement—The research reported here was supported by Ministero Pubblica Istruzione, Italia.

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APPENDIX A

Inversion of the function s(t)

To invert the function

$$s(t) = \int_0^t U \, dt \tag{A.1}$$

we consider U expressed by Taylor series:

$$s(t) = \sum_{j=m}^{\infty} a_j t^j \tag{A.2}$$

where the a_j are known coefficients.

The inverse function $t(s)$ can be written as

$$t(s) = \sum_{i=1}^{\infty} A_i s^{i/m}. \tag{A.3}$$

The function $t(s)$ is expressed by a Taylor series in terms of the variables s only for $m = 1$, i.e. for $U(0) \neq 0$.

From equations (A.2) and (A.3), evaluated for $t \to 0$, one has $A_1^m = 1/a_m$.

To evaluate all the coefficients A_i, one recalls that $s'(t)t'(s) = 1$; i.e. one has

$$1 = \sum_{j=m}^{\infty} j a_j t^{j-1} \sum_{i=1}^{\infty} (i/m) A_i \left(\sum a_j t^j\right)^{i/m-1}.$$

This equation gives successively the unknowns A_i.

APPENDIX B

Some functions related to the error function

$$\operatorname{erf} x = (2/\pi^{1/2}) \int_0^x \exp(-x^2) \, dx;$$

$$\operatorname{erfc} x = 1 - \operatorname{erf} x; \quad a_i = \int_0^\infty \operatorname{erfc}^i x \, dx$$

$$\pi^{1/2} a_1 = 1; \quad \pi^{1/2} a_2 = 2 - 2^{1/2};$$

$$\pi^{3/2} a_3 = \pi - 8^{1/2} \operatorname{arctg} 2^{1/2}$$

$$\pi^{3/2} a_4 = 4\pi(1 + 18^{1/2}) - 48 \cdot 2^{1/2} \operatorname{arctg} 2^{1/2}$$

$$b_{ij}(k) = \int_0^\infty \operatorname{erf}^i x \operatorname{erfc} k^j x \, dx$$

$$a^2 = k^2 + 1; \quad A^2 = a^2/k^2$$

$$\pi^{1/2} b_{11} = A;$$

$$\pi^{3/2} b_{12} = 4(A \operatorname{arctg} A - \operatorname{arctg} k2^{1/2}/k2^{1/2}) - \pi$$

$$\pi^{3/2} b_{21} = (2A - 1 - 2^{1/2})\pi - 4A \operatorname{arctg} A + 8^{1/2} \operatorname{arctg}(k/2^{1/2})$$

$$\begin{aligned}\pi^{3/2} b_{13} = {} & \pi^{3/2}(a_3 - 3a_1)/k - \pi(1 + 6A) \\ & + 12A[\operatorname{arctg} A + \operatorname{arctg}(1 + 2/A^2)^{1/2}] \\ & - (\operatorname{arctg} 2^{1/2}/2)/2^{1/2} \\ & + \operatorname{arctg}[(A^2 + 1)/2^{1/2} + (A^2 + 2)k2^{1/2}]/2^{1/2}.\end{aligned}$$

COUCHES LIMITES VARIABLES DE QUANTITE DE MOUVEMENT ET D'ENERGIE POUR L'ECOULEMENT LAMINAIRE SUR PLAQUE PLANE

Résumé—Une solution analytique approchée est présentée pour les champs de vitesse et de température dans un écoulement laminaire sur plaque plane. Le fluide et la plaque sont initialement au repos et isothermes. La plaque est brusquement mise en vitesse constante, la température de paroi est maintenue constante à une valeur différente de l'initiale. Les équations de couche limite sont résolues en extension d'une méthode développée antérieurement par l'un des auteurs et basée sur l'expression des profils de vitesses et de températures inconnues par des séries de Taylor d'une variable convenable et en utilisant des relations différentielles et intégrales obtenues à partir des équations de bilan. La solution, obtenue sous forme analytique pour le premier ordre d'approximation, montre un bon accord avec les données numériques connues. La convergence de la méthode a été testée dans quelques cas particuliers en considérant des ordres d'approximation plus élevés dans les formules de Taylor: l'approximation du troisième ordre donne des résultats pratiquement égaux à ceux de la solution exacte.

INSTATIONÄRE IMPULS- UND ENERGIE-GRENZSCHICHTEN BEI DER LAMINAREN STRÖMUNG ÜBER EINE EBENE PLATTE

Zusammenfassung—Es wird eine angenäherte analytische Lösung für instationäre Geschwindigkeits- und Temperaturfelder bei laminarer Plattenströmung vorgestellt. Am Anfang sind das Fluid und die Platte in Ruhe und isotherm. Die Platte wird rasch mit konstanter Geschwindigkeit in Bewegung versetzt, die Wandtemperatur wird auf einem konstanten Wert gehalten, der sich von dem Anfangswert unterscheidet. Die Grenzschichtgleichungen wurden durch Erweiterung einer zuvor von einem der Autoren entwickelten Methode gelöst. Das Verfahren beruht darauf, daß die unbekannten Geschwindigkeits- und Temperaturprofile mit Hilfe einer geeigneten Variablen durch eine Taylor-Reihe dargestellt werden und Differential- und Integral-Bedingungen Verwendung finden, welche sich aus den Gleichgewichtsbedingungen ergeben. Die Lösung, in geschlossener Form für die Approximation erster Ordnung, liefert eine gute Übereinstimmung mit den in der Literatur erhältlichen numerischen Daten. Die Konvergenz des Verfahrens wurde für einige besondere Fälle durch Anwendung von Approximationen höherer Ordnung bei der Taylor-Entwicklung untersucht: Die dritte Approximation liefert Ergebnisse, die nahezu mit den exakten Resultaten übereinstimmen.

НЕСТАЦИОНАРНЫЕ ДИНАМИЧЕСКИЙ И ТЕПЛОВОЙ ПОГРАНИЧНЫЕ СЛОИ ПРИ ЛАМИНАРНОМ ОБТЕКАНИИ ПЛОСКОЙ ПЛАСТИНЫ

Аннотация—Представлено приближенное аналитическое решение для нестационарных полей скорости и температуры при ламинарном обтекании плоской пластины. Вначале жидкость и пластина находятся в состоянии покоя и являются изотермическими. Пластина внезапно приводится в движение с постоянной скоростью, температура поддерживается постоянной, но отличной от начальной. Уравнения пограничного слоя решены с помощью нового варианта метода, предложенного ранее одним из авторов, в основе которого лежит определение профилей скорости и температуры с помощи формулы Тейлора с соответствующей переменной и использование дифференциальных и интегральных соотношений, полученных из балансных уравнений. Решение, полученное в замкнутом виде для аппроксимации первого порядка, согласуется с имеющимися в литературе численным данными. Сходимость метода проверена на ряде частных случаев путем рассмотрения аппроксимаций высшего порядка в формуле Тейлора: аппроксимация третьего порядка дает результаты, практически совпадающие с точными.